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1984 J. Phys. A: Math. Gen. 17 1385

(http://iopscience.iop.org/0305-4470/17/6/033)

J. Phys. A: Math. Gen. 17 (1984) 1385–1386. Printed in Great Britain

COMMENT

On the potential due to a circular parallel plate capacitor

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Received 21 September 1983

Abstract. A recently proposed analytic solution to a celebrated problem in classical potential theory is shown to be incorrect.

The problem of determining the exact solution for the potential due to a circular parallel plate capacitor is a celebrated one (Sneddon 1966), with the most successful discussion to date being that by Love (1949), in which the mathematical problem is recast in terms of a Fredholm integral equation of the second kind over a finite domain.

Recently, Atkinson *et al* (1983) have proposed a new solution for the potential Φ, based on the ansatz

$$\Phi(\rho, z) = \int_0^\infty A(k)[\exp(-k|z-L|) - \exp(-k|z+L|)] J_0(\rho k)\, dk \tag{1}$$

where ρ and z are cylindrical coordinates, with the plates defined by $z = L$, $\rho < a$ and $z = -L$, $\rho < a$; J_0 denotes the usual Bessel function; and $A(k)$ is a function to be determined by the boundary conditions

$$\Phi(\rho, \pm L) = \pm V, \qquad 0 < \rho < a. \tag{2}$$

They observe that if one writes

$$A(k) = (2V/\pi)[1 - \exp(-2kL)]^{-1}(\sin ka)/k, \tag{3}$$

then the ansatz (1) satisfies the boundary conditions, and they claim that (1) and (3) constitute an analytic solution of the problem.

The error in their analysis lies in the ansatz chosen. A valid solution of the problem is a function $\Phi(\rho, z)$ which satisfies Laplace's equation *everywhere* save on the surface of each plate. The ansatz certainly satisfies Laplace's equation in each of the regions $z > L$, $-L < z < L$ and $z < -L$, but is not even continuously differentiable with respect to z when $z = \pm L$ and $\rho > a$ unless an integral constraint is satisfied. For example consider $z = L$. We have

$$\Phi(\rho, z) = \int_0^\infty A(k)\{\exp[-k(z-L)] - \exp[-k(z+L)]\} J_0(\rho k)\, dk, \qquad z > L \tag{4}$$

$$\Phi(\rho, z) = \int_0^\infty A(k)\{\exp[-k(L-z)] - \exp[-k(z+L)]\} J_0(\rho k)\, dk, \qquad -L < z < L \tag{5}$$

so that

$$\lim_{z\to L^+} \frac{\partial\Phi}{\partial z} - \lim_{z\to L^-} \frac{\partial\Phi}{\partial z} = -2\int_0^\infty kA(k)J_0(\rho k)\,\mathrm{d}k. \tag{6}$$

Since the integral in (6) does not vanish for all $\rho > a$ when $A(k)$ is given by (3) (except in the degenerate limit $L \to \infty$), $\partial\Phi/\partial z$ is discontinuous.

The reduction of the circular parallel plate capacitor problem to quadratures remains an open problem.

References

Atkinson W J, Young J H and Brezovich I A 1983 *J. Phys. A: Math. Gen.* **16** 2837–41
Love E R 1949 *Q.J. Mech. Appl. Math.* **2** 428–51
Sneddon I N 1966 *Mixed Boundary Value Problems in Potential Theory* (Amsterdam: North-Holland)